



AD FALCON API Manual

Fully Coupled Dynamic Analysis with FALCON

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Contents

1	Fully Coupled Dynamic Analysis with FALCON	3
1.1	Introduction	3
1.2	Conservation of Mass (see Schrefler and Scotta (2001))	3
1.3	Linear Momentum Balance	3
1.4	Balance of Energy	3
1.5	Effective Stress	4
1.6	Work-Conjugate Pairs and FEM Formulation	4
1.6.1	Internal-Energy Rate	4
1.7	Weak Forms	5
1.7.1	Fluid mass balance	5
1.7.2	Generalized Darcy law (see Zienkiewicz et al. (1990))	5
1.7.3	Solid momentum balance	5
1.8	FE Discrete Equations	5
1.8.1	Reduced System: Negligible Fluid-Phase Inertia ($M_w \approx 0$, $M_g \approx 0$)	5
1.9	Appendix A: Finite-Element Matrices and Vectors	6
1.9.1	Mass matrices	6
1.9.2	Coupling matrices	6
1.9.3	Hydraulic conductivity matrices	6
1.9.4	Compressibility matrices	6
1.9.5	Coefficient matrices	7
1.9.6	Load and flow vectors	7
1.10	Table 2: Variable Definitions	7
1.11	References	8

1 Fully Coupled Dynamic Analysis with FALCON

1.1 Introduction

This reference summarises the governing equations for unsaturated-soil dynamics—mass, momentum, and energy balances coupled with Darcy flow and Bishop's effective stress. All symbols used below are listed once in **Table 2**.

1.2 Conservation of Mass (see Schrefler and Scotta (2001))

For each phase $\alpha \in \{s, w, g\}$

$$\frac{\partial(\rho_\alpha n^\alpha)}{\partial t} + \nabla \cdot (\rho_\alpha n^\alpha \dot{\mathbf{u}}_\alpha) = 0 \quad (1)$$

With $\dot{\mathbf{u}}_s = \dot{\mathbf{u}}$ and Darcy velocity $\dot{\mathbf{w}}^\beta$

$$\frac{Dn}{Dt} = (1 - n) \nabla \cdot \dot{\mathbf{u}} \quad (2)$$

Fluid phases $\beta \in \{w, g\}$

$$\frac{n S_\beta}{K_\beta} \frac{Dp_\beta}{Dt} + n \frac{DS_\beta}{Dt} + S_\beta \nabla \cdot \dot{\mathbf{u}} + \nabla \cdot \dot{\mathbf{w}}^\beta + \dot{\mathbf{w}}^\beta \cdot \frac{\nabla \rho_\beta}{\rho_\beta} = 0 \quad (3)$$

1.3 Linear Momentum Balance

$$\nabla \cdot \boldsymbol{\sigma} + \rho \mathbf{b} = \rho \ddot{\mathbf{u}} + \rho^g \ddot{\mathbf{u}}_{ws} + \rho^w \ddot{\mathbf{u}}_{gs} \quad (4)$$

1.4 Balance of Energy

$$\frac{DE_I}{Dt} = \mathcal{P} - \frac{DE_k}{Dt} \quad (5)$$

$$\frac{DE_k}{Dt} = \sum_\alpha \int_\Omega \rho^\alpha \ddot{\mathbf{u}}_\alpha \cdot \dot{\mathbf{u}}_\alpha d\Omega \quad (6)$$

$$\mathcal{P} = \sum_\alpha \left[\int_\Gamma \mathbf{t}^\alpha \cdot \dot{\mathbf{u}}_\alpha d\Gamma + \int_\Omega (\rho^\alpha \mathbf{b} + \mathbf{I}^\alpha) \cdot \dot{\mathbf{u}}_\alpha d\Omega \right] \quad (7)$$

1.5 Effective Stress

Unsaturated effective stress is evaluated with an α -**enriched Bishop formulation** (see Ghorbani and Kodikara (2024)):

$$\boldsymbol{\sigma}' = \boldsymbol{\sigma} + \chi p_w \mathbf{I} + (1 - \chi) p_g \mathbf{I} \quad (8)$$

with $\chi = \alpha_{\text{eff}} S_w$

Table 1: Bishop Effective Stress Formulations

Variant	Expression	Notes
α - enriched (FALCON default)	$\alpha_{\text{eff}}(S_w) = S_w^{\frac{\beta_1}{\beta_2}-1}$, $\chi = S_w^{\frac{\beta_1}{\beta_2}}$	parameters β_1, β_2 tune factors such as the decline of capillary contribution toward dryness

Key properties

- * $\chi \rightarrow 1$ at saturation, $\chi \rightarrow 0$ at full dryness.
- * Choosing $\beta_1 = 1$ and $\beta_2 = 0$ yields $\chi = S_w$ (classical Bishop).
- * The relationship between χ and S_w is assumed independent of hydraulic hysteresis.

These compact relations replace the lengthy derivations while retaining the essential variables for constitutive modelling.

1.6 Work-Conjugate Pairs and FEM Formulation

1.6.1 Internal-Energy Rate

$$\frac{De_I}{Dt} = \boldsymbol{\sigma}' : \nabla \dot{\mathbf{u}} + \sum_{\beta=w,g} n \alpha_{\text{eff}}^{\beta} p_{\beta} \dot{S}_{\beta} + R \quad (9)$$

$$R = \sum_{\beta=w,g} p_{\beta} \left[\frac{n \alpha_{\text{eff}}^{\beta} S_{\beta}}{K_{\beta}} \frac{Dp_{\beta}}{Dt} + \alpha_{\text{eff}}^{\beta} \frac{\dot{\mathbf{w}}^{\beta}}{\rho^{\beta}} \cdot \nabla \rho^{\beta} \right] \quad (10)$$

Neglecting compressibility and convective terms

$$\frac{De_I}{Dt} = \boldsymbol{\sigma}' : \nabla \dot{\mathbf{u}} - n \alpha_{\text{eff}}^w p_c \dot{S}_w \quad (11)$$

1.7 Weak Forms

1.7.1 Fluid mass balance

$$\int_{\Omega} \delta p_w A_w d\Omega + \int_{\Omega} \delta p_g A_g d\Omega + \int_{\Gamma_{qw}} \delta p_w (\dot{\mathbf{w}}^w \cdot \mathbf{n}^* - \bar{\mathbf{w}}^w) d\Gamma - \int_{\Gamma_{qg}} \delta p_g (\dot{\mathbf{w}}^g \cdot \mathbf{n}^* - \bar{\mathbf{w}}^g) d\Gamma = 0 \quad (13)$$

with

$$A_{\beta} = \frac{n S_{\beta}}{K_{\beta}} \frac{D p_{\beta}}{D t} + n \frac{D S_{\beta}}{D t} + S_{\beta} \nabla \cdot \dot{\mathbf{u}} + \nabla \cdot \dot{\mathbf{w}}^{\beta} + \dot{\mathbf{w}}^{\beta} \cdot \frac{\nabla \rho_{\beta}}{\rho_{\beta}} \quad (14)$$

1.7.2 Generalized Darcy law (see Zienkiewicz et al. (1990))

$$\dot{\mathbf{w}}^{\beta} = \mathbf{k}_{\beta} [-\nabla p_{\beta} + \rho_{\beta}(\mathbf{b} - \ddot{\mathbf{u}})], \quad \mathbf{k}_{\beta} = \mathbf{k}_{\text{int}} k_{r\beta} / \eta_{\beta} \quad (15)$$

1.7.3 Solid momentum balance

$$\int_{\Omega} \delta \mathbf{u}^T (\mathbf{L}^T \boldsymbol{\sigma} + \rho \mathbf{b} - \rho \ddot{\mathbf{u}}) d\Omega + \int_{\Gamma_t} \delta \mathbf{u}^T (\bar{\mathbf{t}} - \bar{\mathbf{l}}_o^T \boldsymbol{\sigma}) d\Gamma = 0 \quad (16)$$

1.8 FE Discrete Equations

Unknown vectors: $\mathbf{U}, \mathbf{P}_w, \mathbf{P}_g$.

$$\mathbf{M}_u \ddot{\mathbf{U}} + \mathbf{C} \dot{\mathbf{U}} + \mathbf{K} \mathbf{U} - \mathbf{Q}_w^* \mathbf{P}_w - \mathbf{Q}_g^* \mathbf{P}_g = \mathbf{F}_u \quad (17)$$

$$\mathbf{M}_w \ddot{\mathbf{U}} + \mathbf{Q}_w^T \dot{\mathbf{U}} + \mathbf{C}_{ww} \dot{\mathbf{P}}_w + \mathbf{C}_{wg} \dot{\mathbf{P}}_g + \mathbf{H}_{ww} \mathbf{P}_w = \mathbf{F}_w \quad (18)$$

$$\mathbf{M}_g \ddot{\mathbf{U}} + \mathbf{R}_g^T \dot{\mathbf{U}} + \mathbf{C}_{gw} \dot{\mathbf{P}}_w + \mathbf{C}_{gg} \dot{\mathbf{P}}_g + \mathbf{H}_{gg} \mathbf{P}_g = \mathbf{F}_g \quad (19)$$

(See Appendix A for matrix definitions.)

1.8.1 Reduced System: Negligible Fluid-Phase Inertia ($\mathbf{M}_w \approx 0, \mathbf{M}_g \approx 0$)

In many geomechanics applications the inertia of pore fluids is negligible compared with the solid skeleton. Assuming $\mathbf{M}_w \approx 0$ and $\mathbf{M}_g \approx 0$, the discrete equations reduce to

$$\mathbf{M}_u \ddot{\mathbf{U}} + \mathbf{C} \dot{\mathbf{U}} + \mathbf{K} \mathbf{U} - \mathbf{Q}_w^* \mathbf{P}_w - \mathbf{Q}_g^* \mathbf{P}_g = \mathbf{F}_u \quad (17R)$$

$$\mathbf{Q}_w^T \dot{\mathbf{U}} + \mathbf{C}_{ww} \dot{\mathbf{P}}_w + \mathbf{C}_{wg} \dot{\mathbf{P}}_g + \mathbf{H}_{ww} \mathbf{P}_w = \mathbf{F}_w \quad (18R)$$

$$\mathbf{R}_g^T \dot{\mathbf{U}} + \mathbf{C}_{gw} \dot{\mathbf{P}}_w + \mathbf{C}_{gg} \dot{\mathbf{P}}_g + \mathbf{H}_{gg} \mathbf{P}_g = \mathbf{F}_g \quad (19R)$$

These reduced equations are used by default in FALCON unless the inclusion of fluid-phase inertia is explicitly required.

1.9 Appendix A: Finite-Element Matrices and Vectors

1.9.1 Mass matrices

$$\mathbf{M}_u = \int_{\Omega} \mathbf{N}_u^T \rho \mathbf{N}_u d\Omega \quad (20)$$

$$\mathbf{M}_w = \int_{\Omega} (\nabla \mathbf{N}_{pw})^T (\mathbf{k}_g \rho_g + \mathbf{k}_w \rho_w) \mathbf{N}_u d\Omega \quad (21)$$

$$\mathbf{M}_g = \int_{\Omega} (\nabla \mathbf{N}_{pg})^T \mathbf{k}_g \rho_g \mathbf{N}_u d\Omega \quad (22)$$

1.9.2 Coupling matrices

At the element level, the coupling matrices that appear in Eqs. (17)–(19) are assembled from Gauss-point contributions and can be written in compact form as

$$\mathbf{Q}_w^* = \int_{\Omega} \mathbf{B}^T (\chi \mathbf{m} - \mathbf{S}_{ep}) \mathbf{N}_{pw} d\Omega, \quad \mathbf{Q}_g^* = \int_{\Omega} \mathbf{B}^T ((1 - \chi) \mathbf{m} + \mathbf{S}_{ep}) \mathbf{N}_{pg} d\Omega \quad (23)$$

for the stress-pressure couplings in the solid momentum balance, and

$$\mathbf{Q}_w = \int_{\Omega} (S_w + e \frac{\partial S_w}{\partial e}) \mathbf{B}^T \mathbf{m} \mathbf{N}_{pw} d\Omega, \quad \mathbf{R}_g = \int_{\Omega} (1 - S_w - e \frac{\partial S_w}{\partial e}) \mathbf{B}^T \mathbf{m} \mathbf{N}_{pg} d\Omega \quad (24)$$

for the velocity–pressure couplings that enter the fluid mass balances. Here $\mathbf{m}$ is the volumetric projector (see Table 2), χ is the Bishop-type weighting factor from the effective-stress model, and $\mathbf{S}_{ep}$ is the suction-coupling vector returned by the UMAT interface (see [UMAT development](#) and the [GCC model example](#)).

In the saturated limit with negligible saturation derivatives ($S_w \rightarrow 1$, $\partial S_w / \partial e \rightarrow 0$, $\mathbf{S}_{ep} \rightarrow 0$), these expressions reduce to the classical Biot-type forms based on $\mathbf{B}^T \mathbf{m} \mathbf{N}_{pw}$.

1.9.3 Hydraulic conductivity matrices

$$\mathbf{H}_{ww} = \int_{\Omega} (\nabla \mathbf{N}_{pw})^T \mathbf{k}_w \nabla \mathbf{N}_{pw} d\Omega \quad (25)$$

$$\mathbf{H}_{gg} = \int_{\Omega} (\nabla \mathbf{N}_{pg})^T \mathbf{k}_g \nabla \mathbf{N}_{pg} d\Omega \quad (26)$$

1.9.4 Compressibility matrices

$$\mathbf{C}_{ww} = \int_{\Omega} \mathbf{N}_{pw}^T \mathbf{C}_1^* \mathbf{N}_{pw} d\Omega \quad (27)$$

$$\mathbf{C}_{gg} = \int_{\Omega} \mathbf{N}_{pg}^{\top} \mathbf{C}_2^* \mathbf{N}_{pg} d\Omega \quad (28)$$

$$\mathbf{C}_{wg} = \int_{\Omega} \mathbf{N}_{pw}^{\top} \mathbf{C}_3^* \mathbf{N}_{pg} d\Omega \quad (29)$$

$$\mathbf{C}_{gw} = \int_{\Omega} \mathbf{N}_{pg}^{\top} \mathbf{C}_3^* \mathbf{N}_{pw} d\Omega \quad (30)$$

$$\mathbf{R}_c = \int_{\Omega} \mathbf{B}^{\top} \mathbf{C}_6 \mathbf{m} \mathbf{N}_{pg} d\Omega \quad (31)$$

1.9.5 Coefficient matrices

$$\mathbf{C}_1^* = \frac{nS_w}{K_w} - n \frac{\partial S_w}{\partial p_c} \quad (32)$$

$$\mathbf{C}_2^* = \frac{nS_g}{K_g} - n \frac{\partial S_w}{\partial p_c} \quad (33)$$

$$\mathbf{C}_3^* = n \frac{\partial S_w}{\partial p_c} \quad (34)$$

$$\mathbf{C}_6 = 1 - \left(S_w + e \frac{\partial S_w}{\partial e} \right) \quad (35)$$

1.9.6 Load and flow vectors

$$\mathbf{F}_u = \int_{\Omega} \mathbf{N}_u^{\top} \rho \mathbf{b} d\Omega + \int_{\Gamma_t} \mathbf{N}_u^{\top} \bar{\mathbf{t}} d\Gamma \quad (36)$$

$$\mathbf{F}_w = \int_{\Omega} (\nabla \mathbf{N}_{pw})^{\top} \mathbf{k}_w \rho_w \mathbf{b} d\Omega - \int_{\Gamma_{qw}} \mathbf{N}_{pw}^{\top} \bar{\mathbf{w}}^w d\Gamma \quad (37)$$

$$\mathbf{F}_g = \int_{\Omega} (\nabla \mathbf{N}_{pg})^{\top} \mathbf{k}_g \rho_g \mathbf{b} d\Omega - \int_{\Gamma_{qg}} \mathbf{N}_{pg}^{\top} \bar{\mathbf{w}}^g d\Gamma \quad (38)$$

1.10 Table 2: Variable Definitions

Symbol	Description
$M^{\alpha}, \Omega^{\alpha}$	mass, volume of phase $\alpha \in \{s, w, g\}$
$\rho_{\alpha}, \rho^{\alpha}, \rho$	phase density, partial density, mixture density
n^{α}, n	phase volume fraction, porosity
S_w, S_g	water and air degree of saturation
$p_w, p_g, p_c = p_g - p_w$	pore-water pressure, pore-air pressure, suction

Symbol	Description
σ, σ'	total stress, effective stress
σ_{net}	net stress = $\sigma + p_g \mathbf{I}$
$\chi = \alpha_{\text{eff}} S_w$	effective-stress weighting factor
$\alpha_{\text{eff}}(S_w)$	saturation correction, = $S_w^{\frac{\beta_1}{\beta_2}-1}$
β_1, β_2	material parameters in α_{eff}
K_β, η_β	bulk modulus and viscosity of phase β
$\mathbf{k}_{\text{int}}, k_{r\beta}, \mathbf{k}_\beta$	intrinsic, relative, and effective permeability
$\mathbf{u}, \dot{\mathbf{u}}, \ddot{\mathbf{u}}$	displacement, velocity, acceleration (solid)
$\mathbf{w}^\beta$	Darcy velocity of phase β
ε	strain tensor
$\mathbf{b}$	body-force vector
$E_I, E_k, \mathcal{P}$	internal energy, kinetic energy, power
$\mathbf{N}, \mathbf{B}, \mathbf{L}$	shape, strain-displacement, differential operators
$\mathbf{M}, \mathbf{K}, \mathbf{C}, \mathbf{H}$	global mass, stiffness, damping/coupling, conductivity matrices
$\mathbf{Q}, \mathbf{R}$	solid-fluid coupling matrices
$\mathbf{M}_u, \mathbf{M}_w, \mathbf{M}_g$	phase-assembled mass matrices
$\mathbf{H}_{ww}, \mathbf{H}_{gg}$	hydraulic conductivity matrices
$\mathbf{C}_{ww}, \mathbf{C}_{gg}, \mathbf{C}_{wg}, \mathbf{C}_{gw}$	compressibility/coupling matrices
$\mathbf{C}_1^*, \mathbf{C}_2^*, \mathbf{C}_3^*, \mathbf{C}_6$	element-level coefficients (see Appendix A)
$\mathbf{n}^*, \mathbf{I}_\sigma^\top$	outward unit normal, stress-traction operator
$\bar{\mathbf{t}}$	prescribed traction (Neumann BC)
$\bar{\mathbf{w}}^\beta$	prescribed Darcy flux of phase β
$\mathbf{U}, \mathbf{P}_w, \mathbf{P}_g$	global DOF vectors (displacement, p_w, p_g)
$\mathbf{F}_u, \mathbf{F}_w, \mathbf{F}_g$	global load / flow vectors
$\mathbf{K}_{NCi}, \mathbf{F}_{NCi}$	contact stiffness and force (segment i)
e	void ratio
$\mathbf{m}$	Coupling vector = $[1, 1, 1, 0, 0, 0]^\top$ in 3D Voigt notation

1.11 References

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