AD FALCON API Manual

Isotropic Linear Elasticity

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1 Isotropic Linear Elasticity

Isotropic linear elastic constitutive model implemented in FALCON.

1.1 Syntax

This model is configured in `% Materials` using `@UMAT:` with category `Mechanical` (for example, `LinearElasticUMAT`).

```
@UMAT: /path/to/LinearElasticUMAT.cpp /path/to/LinearElasticUMAT.hpp
Mechanical \
    YoungsModulus=<E> PoissonsRatio=<nu>
```

To approximate a rigid solid, use a very large `YoungsModulus` and an appropriate `PoissonsRatio`.

See Material Models: Syntax & Conventions for shared rules (directive formatting, spacing, etc.).

1.2 Material parameters

Symbol	Keyword in input	Units	Required	Description
E	`YoungsModulus`	stress	✓	Young's modulus.
ν	`PoissonsRatio`	–	✓	Poisson's ratio.

1.3 Constitutive relation

For an **isotropic** linear elastic material, the mechanical properties are identical in all directions. The constitutive relation is expressed in Voigt notation (order: $[11, 22, 33, 23, 13, 12]$) as:

$$\boldsymbol{\sigma} = \mathbf{C}\boldsymbol{\varepsilon} \tag{1}$$

where $\boldsymbol{\sigma}$ is the stress vector, $\boldsymbol{\varepsilon}$ is the strain vector, and $\mathbf{C}$ is the 6×6 stiffness matrix.

1.3.1 Elastic moduli

The elastic behavior is characterized by two independent material constants. The **bulk modulus** K and **shear modulus** G are related to Young's modulus E and Poisson's ratio ν by:

$$K = \frac{E}{3(1-2\nu)} \tag{2}$$

$$G = \frac{E}{2(1+\nu)} \tag{3}$$

1.3.2 Compliance matrix

The compliance matrix $\mathbf{S} = \mathbf{C}^{-1}$ relates strain to stress:

$$\mathbf{S} = \frac{1}{E}\begin{bmatrix} 1 & -\nu & -\nu & 0 & 0 & 0 \\ -\nu & 1 & -\nu & 0 & 0 & 0 \\ -\nu & -\nu & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2(1+\nu) & 0 & 0 \\ 0 & 0 & 0 & 0 & 2(1+\nu) & 0 \\ 0 & 0 & 0 & 0 & 0 & 2(1+\nu) \end{bmatrix} \tag{4}$$

The shear components appear as $2(1+\nu)$ due to the engineering shear strain convention in Voigt notation (where $\gamma_{ij} = 2\varepsilon_{ij}$ for $i \neq j$).

1.3.3 Stiffness matrix

The stiffness matrix $\mathbf{C}$ is the inverse of the compliance matrix. In terms of bulk modulus K and shear modulus G:

$$\mathbf{C} = \begin{bmatrix} K+\frac{4}{3}G & K-\frac{2}{3}G & K-\frac{2}{3}G & 0 & 0 & 0 \\ K-\frac{2}{3}G & K+\frac{4}{3}G & K-\frac{2}{3}G & 0 & 0 & 0 \\ K-\frac{2}{3}G & K-\frac{2}{3}G & K+\frac{4}{3}G & 0 & 0 & 0 \\ 0 & 0 & 0 & G & 0 & 0 \\ 0 & 0 & 0 & 0 & G & 0 \\ 0 & 0 & 0 & 0 & 0 & G \end{bmatrix} \tag{5}$$

Alternatively, the stiffness matrix can be expressed in terms of the **Lamé parameters** λ and μ:

$$\mathbf{C} = \begin{bmatrix} \lambda+2\mu & \lambda & \lambda & 0 & 0 & 0 \\ \lambda & \lambda+2\mu & \lambda & 0 & 0 & 0 \\ \lambda & \lambda & \lambda+2\mu & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu \end{bmatrix} \tag{6}$$

where: - $\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}$ (first Lamé parameter) - $\mu = G = \frac{E}{2(1+\nu)}$ (second Lamé parameter, equal to the shear modulus)

The relationship between the different representations is: - $K = \lambda + \frac{2}{3}\mu$ - $G = \mu$

1.4 Stress–strain relation

The incremental stress–strain relation is:

$$\Delta\boldsymbol{\sigma} = \mathbf{C}\Delta\boldsymbol{\varepsilon} \tag{7}$$

where $\Delta\boldsymbol{\sigma}$ is the stress increment and $\Delta\boldsymbol{\varepsilon}$ is the strain increment.

1.4.1 Decomposition into volumetric and deviatoric parts

The stress and strain can be decomposed into volumetric (mean) and deviatoric parts:

Volumetric components:

$$\sigma_m = \frac{1}{3}(\sigma_{11} + \sigma_{22} + \sigma_{33}), \quad \varepsilon_v = \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}$$

Deviatoric components:

$$\mathbf{s} = \boldsymbol{\sigma} - \sigma_m \mathbf{I}, \quad \mathbf{e} = \boldsymbol{\varepsilon} - \frac{1}{3}\varepsilon_v \mathbf{I}$$

The constitutive relation separates into: - **Volumetric response**: $\sigma_m = K\varepsilon_v$ - **Deviatoric response**: $\mathbf{s} = 2G\mathbf{e}$

1.5 Parameter constraints

For the material to be physically admissible:

1. **Positive moduli**: Young's modulus must be positive:

$$E > 0$$

2. **Poisson's ratio bounds**: The Poisson's ratio must satisfy:

$$-1 < \nu < 0.5$$

3. **Stability**: For thermodynamic stability, the bulk modulus and shear modulus must be positive:

$$K > 0 \quad \Rightarrow \quad \nu < 0.5$$

$$G > 0 \quad \Rightarrow \quad \nu > -1$$

1.6 Plane strain and plane stress

1.6.1 Plane strain

For plane strain conditions (e.g., $\varepsilon_{33} = 0$), the stress–strain relation reduces to a 3×3 system for $[\sigma_{11}, \sigma_{22}, \sigma_{12}]$ and $[\varepsilon_{11}, \varepsilon_{22}, \gamma_{12}]$:

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \gamma_{12} \end{bmatrix} \tag{8}$$

The out-of-plane stress is:

$$\sigma_{33} = \nu(\sigma_{11} + \sigma_{22})$$

1.6.2 Plane stress

For plane stress conditions (e.g., $\sigma_{33} = 0$), the stress–strain relation becomes:

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \gamma_{12} \end{bmatrix} \tag{9}$$

The out-of-plane strain is:

$$\varepsilon_{33} = -\frac{\nu}{1-\nu}(\varepsilon_{11} + \varepsilon_{22})$$

1.7 Special cases

1.7.1 Nearly incompressible material ($\nu \to 0.5$)

As $\nu \to 0.5$, $K \to \infty$ and the response becomes purely deviatoric ($\varepsilon_v = 0$). Values of ν very close to 0.5 can cause volumetric locking in finite element analyses.

1.8 FALCON mini

The packaged mini tool id is `LinearElastic`. It lives under `mini_tools/LinearElastic` and is input-file-driven.

Bundled case families:

- drained triaxial: `cases/drained/input.txt`
- undrained triaxial: `cases/undrained/input.txt`

Typical CLI usage:

```
falcon --mini-root /path/to/UMATLIB_FALCON/falcon_minis --mini-tool
LinearElastic --mini-input /path/to/case_dir
```

The packaged driver writes `stress_results.csv` for each case.

1.9 Applications and limitations

- Best suited to baseline elastic benchmarks, verification problems, and materials where irreversible deformation is not required.
- Can be used as the mechanical law in uncoupled, coupled, or fully coupled analyses when the remaining hydraulic components are defined separately.
- Does not include plasticity, hardening, softening, suction dependence, or cyclic accumulation mechanisms.

1.10 References (selection)

- Timoshenko, S. P. & Goodier, J. N. (1970). *Theory of Elasticity* (3rd ed.). McGraw-Hill.
- Gurtin, M. E. (1981). *An Introduction to Continuum Mechanics*. Academic Press.
- Bower, A. F. (2009). *Applied Mechanics of Solids*. CRC Press.